VETOR

Asymptotic Homogenization and Fractional Calculus Applied to Micro-heterogeneous Media Modelling: an Introduction with the Case of a Microperiodic and Linear Functionally Graded Rod☆

Homogeneização Assintótica e Cálculo Fracionário na Modelagem de Meios Micro-heterogêneos: uma Introdução com o Caso de uma Barra Funcionalmente Graduada, Microperiódica e Linear

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Abstract

The study of materials with complex structure, like the functionally graded, is a field of increasing interest, what happens mostly because the importance of these materials in the industry. In this work, the Asymptotic Homogenization Method and Fractional Calculus are both applied in a problem which models the behaviour of a micro-heterogeneous material, like the functionally graded. The goal of this work is the study of the association possibilities between these two tools, since which one are providing important results in the mathematical modelling of complex structures. The results show that each methodology reproduce a different aspect of the phenomenon: the Homogenization stays in the microstructure details and the fractional derivative takes care of a macroscopic behaviour, which nature is possibly dissipative. Here are important information, but a deeper and more diverse approach is necessary to provide strong e more general statements about this theme.

Keywords

Asymptotic Homogenization • Fractional Calculus • Conformable Derivatives • Functionally Graduated Materials

Resumo

O estudo de materiais com estrutura complexa, como os funcionalmente graduados, tem cada vez mais chamado a atenção, seja pela dificuldade em obter os resultados ou pela importância de tais materiais em diversos ramos da indústria. Neste trabalho, o Método de Homogeneização Assintótica e ferramentas do Cálculo Fracionário são aplicados para modelar o comportamento um material micro-heterogêneo, como os funcionalmente graduados. O interesse principal desse trabalho é encontrar uma forma de associar ambas metodologias, que têm fornecido bons resultados quando aplicadas em problemas envolvendo estruturas complexas, mas de forma separada. Os resultados

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obtidos mostram que cada metodologia reproduz diferentes aspectos do fenômeno: a Homogenização está nos detalhes da microestrutura, enquanto que a derivada fracionária se ocupa de um comportamento macroscópico, cuja natureza pode ser dissipativa. Aqui estão resultados importantes, porém uma abordagem mais profunda e diversificada é necessária a fim de fornecer conclusões mais fortes e generalizadas acerca do tema.

Palavras-chave

Homogeneização Assintótica • Cálculo Fracionário • Derivadas Compatíveis • Materiais Funcionalmente Graduados

1 Introduction

Functionally Graded Materials (FGM) are materials which properties show spatial smooth variation [1], allowing these materials present a peculiar behaviour in some situations, like thermal resistance or structural integrity [2]. Thereby, they have been employed in several areas of the industry, for example: aerospace, maritime, automotive and biological [3]. The FGM's microstructure (the structure in the heterogeneity scale) has an important roll in its physical macroscopic behaviour. In fact, this behaviour depends on the phenomena which occurs in the microscopic scale [4]. Thus, the knowledge about the existing relations between the microstructure and the physical behaviour of the material is very important to have.

In this work are considered the FGM which microscale is much smaller than the macro, and much bigger than the atomic scale, simultaneously. For these ones, the differential equation problems involved in its behaviour modelling present coefficients with rapidly oscillation. Because of that, harder is the direct application of numerical methods in this type of problems [5], while thin the microscale is, a thinner mesh is necessary. However, at the same time, the equivalent homogeneity hypothesis is satisfied, so the micro-heterogeneous material can be considered equivalent to a homogeneous one. For this ideal homogeneous material, the problem involved has constant coefficients, make this problem a lot easier to solve (in the sense of computational cost). Besides that, the solution obtained from this "homogeneous problem" is sufficiently close to the solution from the "original problem" (for the micro-heterogeneous material). The process to obtain these homogeneous material is called of Homogenization.

Among the homogenization methods, we can cite the Asymptotic Homogenization Method (AHM) [6], which considers an asymptotic approximation of the original problem solution, in the form of a potential series for a small parameter ε, with double-scale (macro and micro). There are several advantages from this methodology, but these two are the more interesting ones: low computational cost with the numerical methods application; and the obtaining of good approximations for the original solution. The AHM has two important applications: approximate the original problem solution by asymptotic estimates [7] and determine the effective behaviour of the micro-heterogeneous material using the equivalent homogeneous one [8].

Another tool that has been very used in the mathematical modelling field is the Fractional Calculus (FC), a theory based in the generalization of the concepts from the usual Differential-Integral Calculus, where the operators of integer order are replaced by the ones with non-integer order [9]. Among others applications, stands out that this tool has been useful to reproduces behaviour with dissipation nature, in the solutions of so many mathematical problems. In [10], by using a operator of Caputo type, the gravity effect are reproduced in a harmonic oscillator problem - without considering gravity in the equation; the same effect can be verified in [11], where this operator is used to evaluate the thermal distribution in a thin rectangular plate.

The fractional operators more classical are the Riemann-Liouville and Caputo type[12], but these ones (and the most of the nonlocal operators) have a peculiar theory behind. For example, a lot of the properties from the usual Calculus (like the chain rule) are not valid for these operators. The computational cost that is necessary in its applications is expensive too, in relation to local operators [13]. In this context, in [14] are introduced the concept of Conformable Derivative, which the main advantage is hold important properties from the usual Calculus (like the chain rule - different of the operators above mentioned). This Conformable operator can't be considered exactly a fractional one, most for being a local operator. But, has been shown that is valid considering these ones in the application context of the FC, most because of the results achieved with these local operators, what are very alike to those obtained with the non-local operators [13, 15], in certain situations.

Given the above, the purpose of this paper is to apply the AHM and the Conformable operators to a boundary value problem (BVP) for the steady-state linear diffusion equation, and evaluate the results obtained from this two methodologies. A continuously differentiable coefficient, ε-periodic, will be taken to simulate a functionally graded rod.

2 Methodology

2.1 BVP formulation

The functionally graded rod can be idealized by the one-dimensional case for the problem, represented by $[0,l] \subset \mathbb{R}$. As the structure has a microperiodic property (ε-periodic by the way, where $\varepsilon \ll 1$), the periodicity cell will be $[0,\varepsilon]$. The scale separation will be taken like that: x represents the macroscale, and $y = x/\varepsilon$, the micro one. So, $[0,\varepsilon]$ will be equivalent to the interval $[0,1]$, in the microscale. The functionally graded property will be represented by K^{ε}, which varies continuously. In the Fig. 1 follows a illustration of this idea.

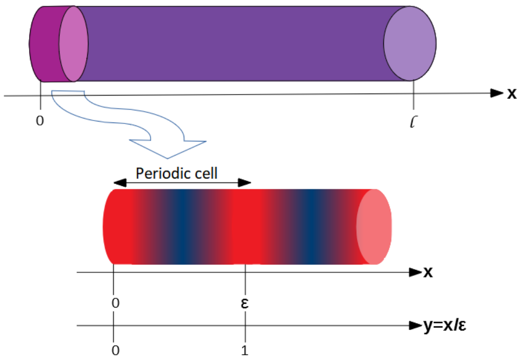

Figure 1: Illustration of a functionally graded and microperiodic bar.

To model the phenomenon of interest, the following BVP for the steady-state linear diffusion equation will be considered:

$$\begin{cases} \dfrac{d}{dx}\left[K^{\varepsilon}(x)\dfrac{du^{\varepsilon}}{dx}\right] = f(x), \; x \in (0,l) \\ u^{\varepsilon}\mid_{x=0} = g_1 \\ u^{\varepsilon}\mid_{x=l} = g_2 \end{cases}, \tag{1}$$

where $K^{\varepsilon} \in C^1([0,l])$, ε-periodic in x, positive and strictly limited, and $f(x) \in C([0,l])$. Under these hypothesis, the particular solution for the BVP in Eq. (1) is:

$$u^{\varepsilon}(x) = \int_0^x \left[\frac{\int_0^s f(t)dt + C_1}{K^{\varepsilon}(s)}\right] ds + g_1, \tag{2}$$

where

$$C_1 = \left[\int_0^l \frac{1}{K^{\varepsilon}(s)} ds\right]^{-1} \left(g_2 - g_1 - \int_0^l \frac{\int_0^s f(t)dt}{K^{\varepsilon}(s)} ds\right). \tag{3}$$

It is important to observe that, by the continuity of $K^{\varepsilon}(x)$ and $f(x)$, the existence of the integrals in Eqs. (2) and (3) is guaranteed.

Furthermore, $u^{\varepsilon}(x)$ can represents the temperature, electric field or a displacement; $K^{\varepsilon}(x)$ the thermal or electric conductivity, or the elasticity coefficient of the media; and $f(x)$ the thermal or electrical sources or even a external force, depending on the context of interest: thermal, electric or mechanic.

2.2 Asymptotic Homogenization Method

Initiating the AHM approach, first is took an asymptotic expansion with double scale, for the BVP in Eq. (1) solution, $u^{\varepsilon}(x)$:

$$u^{\varepsilon}(x) \sim u^{(2)}(x,y) = v_0(x) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y), \tag{4}$$

where $u_k(x,y)$ are 1-periodic for y (for $k = 1,2$). From the application of $u^{(2)}(x,y)$ into the BVP equation in Eq. (1), considering the chain rule in relation to y, is obtained:

$$\begin{aligned} \frac{d}{dx}\left[K^{\varepsilon}(x)\frac{du^{\varepsilon}}{dx}\right] - f(x) \;\; &\approx \;\; \varepsilon^{-1}\left[\frac{\partial}{\partial y}\left(K(y)\left(\frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right)\right)\right] + \\ &+ \;\; \varepsilon^{0}\left[\frac{\partial}{\partial x}\left(K(y)\left(\frac{dv_0}{dx} + \frac{\partial u_1}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(K(y)\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)\right) - f(x)\right] + \\ &+ \;\; O(\varepsilon) \; . \end{aligned} \tag{5}$$

The expansion $u^{(2)}(x,y)$ will be a good approximation of the solution of the BVP Eq. (1), if the factors of the ε potentials which have exponents less than 1 are equal to 0, to get a ε order error. In that case, $u^{(2)}(x,y)$ will be an asymptotic formal solution (AFS) for the BVP in Eq. (1).

In order to satisfy this need, the follows equations have to be considered:

$$\varepsilon^{-1} \quad : \quad \frac{\partial}{\partial y}\left[K(y)\left(\frac{dv_0}{dx}+\frac{\partial u_1}{\partial y}\right)\right]=0, \tag{6}$$

$$\varepsilon^{\ 0} \quad : \quad \frac{\partial}{\partial x}\left[K(y)\left(\frac{dv_0}{dx}+\frac{\partial u_1}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[K(y)\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)\right]-f(x)=0. \tag{7}$$

From the application of $u^{(2)}(x,y)$ in the boundary conditions in the BVP, is obtained:

$$v_0(0)=g_1 \text{ e } v_0(l)=g_2, \tag{8}$$

$$u_1(0,0)=u_1\,(l,0)=0, \tag{9}$$

$$u_2(0,0)=u_2\,(l,0)=0. \tag{10}$$

The equations (6) e (9) make up the BVP for determinate $u_1(x,y)$, as Eqs (7) and (10) for determinate $u_2(x,y)$. Taking x as a fixed parameter, both problems will present this aspect:

$$\begin{cases} \dfrac{d}{dy}\left[K\,(y)\,\dfrac{dN}{dy}\right]=F(y),\ y\in(0,1) \\ N(0)=0 \end{cases}, \tag{11}$$

where $N(y)$ is the 1-periodic solution searched.

The existence and uniqueness of the solution of these type of problems is guaranteed by the Lemma 1 that follows:

Lemma 1 *[6] Let $K(y)$ an 1-periodic, positive, limited and continuously differentiable function in $[0,1]$, and $F(y)$ just continuous and 1-periodic. Thus, a necessary and sufficient condition to existence of a 1-periodic solution $N(y)$ for the equation:*

$$\frac{d}{dy}\left[K\,(y)\,\frac{dN}{dy}\right]=F(y) \tag{12}$$

is $\int_0^1 F dy=0$.

Besides that, $N(y)$ is unique, except for an additive constant, namely $N(y)=\tilde{N}(y)+C$, where $\tilde{N}$ is the 1-periodic solution of (12), such that $\tilde{N}(0)=0$, and C is an arbitrary constant.

The Lemma proof brings some important details for the determination of $N(y)$ and can be consulted in [6].

The necessary condition of the Lemma 1 is naturally satisfied for the $u_1(x,y)$ problem (Eqs. (6) and (9)). And to ensure the existence and uniqueness of $u_2(x,y)$, solution of the BVP formed by Eqs (7) and (10), is necessary to solve the following equation:

$$\hat{K}\frac{d^2v_0}{dx^2}=f(x),\ x\in(0,l)\,, \tag{13}$$

where $\hat{K}=\left[\int_0^1 (K(y))^{-1}\,dy\right]^{-1}$.

The Equation (13), together with the conditions in Eq. (8), represent the BVP for the equivalent homogeneous material. This BVP is the limit of the recurrent sequence of problems for the ε potentials, when $\varepsilon\to 0^+$, or even more, the called Homogenized Problem. In that way, $\hat{K}$ is the homogeneous material property. It is worth to note, that the solution $v_0(x)$ is a good estimate for $u^{\varepsilon}(x)$, presenting an error with ε order [6].

Now, from solve the above problems, the following formulas can be found, for the AFS $u^{(2)}(x,y)$:

$$v_0(x) \;=\; \frac{1}{\widehat{K}}\left[\int_0^x\int_0^s f(t)\,dt\,ds+\frac{x}{l}\left(\widehat{K}\,(g_2-g_1)-\int_0^l\int_0^s f(t)\,dt\,ds\right)\right]+g_1, \tag{14}$$

$$u_1(x,y) \;=\; \frac{dv_0}{dx}\int_0^y\left(\frac{\hat{K}}{K(s)}-1\right)ds, \tag{15}$$

$$u_2(x,y) \;=\; \frac{d^2v_0}{dx^2}\int_0^y\left(\frac{\hat{K}}{K(s)}\int_0^1 N_1(t)dt-N_1(s)\right)ds. \tag{16}$$

2.3 Fractional Calculus: Conformable Derivative and Integral

First of all, the conformable derivative is defined as follows:

Definition 1 *[14] Let $f : [0, \infty) \to \mathbb{R}$ a function, $\alpha \in (0, 1)$ and $t > 0$. Thus, the f conformable derivative of α order, denoted by $T_\alpha (f)$, is defined by:*

$$T_\alpha (f)(t) = \lim_{\delta \to 0} \frac{f(t + \delta t^{1-\delta}) - f(t)}{\delta}. \tag{17}$$

Some observations about the Definition 1: (i) in the case of the limit in Eq.(17) exists, f is α-differentiable called; (ii) to summarize the notation, $f^{(\alpha)}$ or $\frac{d^\alpha f}{dt^\alpha}$ will be take instead $T_\alpha (f)$; (iii) if f is α-diferentiable in $(0, a)$ for some $a \in \mathbb{R}$ and the limit $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, so $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$; (iv) if $\alpha \to 1$ was taken, thus the $f^{(\alpha)}$ coincides with the usual definition for a derivative of order 1.

The main idea of this definition is to maintain the most as possible the integer order derivatives properties. In the Theorem 1, follow some of these ones:

Theorem 1 *[14] Let $\alpha \in (0, 1)$, and f and g α-diferentiable functions for $t > 0$. So:*
(1) $T_\alpha (af + bg) = aT_\alpha (f) + bT_\alpha (g)$, for all $a, b \in \mathbb{R}$;
(2) $T_\alpha (t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$;
(3) $T_\alpha(\lambda) = 0$, for λ constant;
(4) $T_\alpha (fg) = fT_\alpha (g) + T_\alpha (f) g$;
(5) $T_\alpha \left(\frac{f}{g}\right) = \frac{fT_\alpha(g) - T_\alpha(f)g}{g^2}$, if $g(t) \neq 0$;
(6) If f differentiable in the usual sene, so: $T_\alpha (f(t)) = t^{(1-\alpha)} \frac{df}{dt}(t)$.

It's important to note the item (6) from Theorem 1, because this property allows to transform a fractional differential equation into a integer order one, if the space of functions considered is the of differentiable functions.

The concept of conformable integral of a function f, is defined as follows:

Definition 2 *[14] Letf $: [0, \infty) \to \mathbb{R}$ an integrable function (in the usual sense), $\alpha \in (0, 1)$ and $a \geq 0$. Thus, the f conformable integral of α-order, denoted by $I^a_\alpha (f)$, is defined by::*

$$I^a_\alpha (f)(t) := I^a_1 \left(t^{\alpha-1} f\right)(t) = \int_a^t x^{\alpha-1} f(x) dx. \tag{18}$$

The property of linearity is valid for the α-fractional integral, because this one is defined from the Riemann integral, that is linear. Another properties, which are important to solve fractional differential equations with the conformable operator, follow in the Theorems 2 and 3.

Theorem 2 *[14] Let $\alpha \in (0, 1)$, $t \geq a \geq 0$ and f an integrable function in relation toI_α. So,*

$$T_\alpha \left[I^a_\alpha(f)\right](t) = f(t). \tag{19}$$

Theorem 3 *[16] Let $\alpha \in (0, 1)$, $t \in (a, b)$, where $a \geq 0$ and $f : [a, b] \to \mathbb{R}$ a differentiable function. So:*

$$I^a_\alpha \left[T_\alpha(f)\right](t) = f(t) - f(a). \tag{20}$$

The theorem 3 can be considered as a fractional version of the Fundamental Theorem of the Calculus.

In possession of these facts, it's possible to obtain a fractional version of the BVP of interest, in Eq. (1), namely:

$$\frac{d^\alpha}{dx^\alpha} \left[K^\varepsilon (x) \frac{d^\alpha u^\varepsilon}{dx^\alpha}\right] = f(x), \; x \in (0, l), \tag{21}$$

where $\alpha \in (0, 1)$, that its solution can be obtained by the α-fractional integration of the Eq. (21), considering the Theorems 1 and 3, and the boundary conditions in (1):

$$u^\varepsilon(x) = I^0_\alpha \left[\frac{I^0_\alpha(f)(s) + C_1}{K^\varepsilon(s)}\right](x) + g_1, \tag{22}$$

where

$$C_1 = \left[I^0_\alpha \left(\frac{1}{K^\varepsilon(s)}\right)(l)\right]^{-1} \left(g_2 - g_1 - I^0_\alpha \left[\frac{I^0_\alpha(f)(l)}{K^\varepsilon(s)}\right](l)\right). \tag{23}$$

It's possible to see that the aspect of the solution in Eqs (22) and (23) are the same of those in Eqs. (2) and (3), for the BVP in Eq. (1). The practical effects of consider these fractional operators in this BVP will be evaluated in some examples, in the next section.

3 Some numerical examples

The first example to be solved is the following BVP:

$$\begin{cases} \dfrac{d}{dx}\left[K^{\varepsilon}(x)\dfrac{du^{\varepsilon}}{dx}\right] = -1,\ x \in (0,1) \\ u^{\varepsilon}(0) = u^{\varepsilon}(1) = 0 \\ K^{\varepsilon}(x) = 1 + \dfrac{1}{4}\sin\left(2\pi\dfrac{x}{\varepsilon}\right) \end{cases}, \tag{24}$$

where the coefficient $K^{\varepsilon}(x)$ was took like that to attend the hypothesis of continuity and limitation, and ε-periodicity, as is shown in Fig. 2; where is its rapidly oscillating aspect clear too, while ε is decreasing.

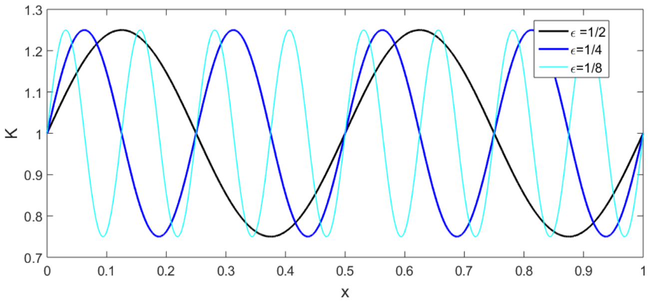

Figure 2: The behaviour of K^{ε} for some values of the small parameter ε.

Executing the AHM methodology, the approximations $v_0(x)$ and $u^{(2)}(x,y)$ are obtained. For v_0, by Eq. (14), taking in account that

$$\hat{K} = \left[\int_0^1 \frac{1}{1+\frac{1}{4}\sin(2\pi y)}dy\right]^{-1} = \frac{\sqrt{15}}{4}, \tag{25}$$

is obtained:

$$v_0(x) = \frac{2}{\sqrt{15}}\left(x - x^2\right). \tag{26}$$

The Fig. 3 brings the comparison of v_0 with the asymptotic approximation $u^{(2)}$ for $\varepsilon = 1/8$. It's possible to check the macroscopic aspect of the first one, and how the second one reproduces the local details of the exact solution u^{ε}. These simulations was executed for $u^{(2)}(x,y)$ by the integrals in Eqs (15) and (16), using the Simpson 1/3 rule [17], with a mesh $h = 0.05\varepsilon$ - in that way considered to the mesh follows the values of ε.

Considering the fractional equation for this example, as in the Eq. (21), the integrals in Eqs. (22) and (23) do not converge. In fact, the solution u^{ε} obtained in that case do not satisfies the boundary conditions of this example. The Fig. 4 shows this fact.

However, in the case of a coefficient $K^{\varepsilon}(x)$ constant, these integrals (Eqs. (22) and (23)) converge, presenting an analytical form. So, seems right take in the place of $K^{\varepsilon}(x)$, the effective coefficient $\hat{K}$, which represents the property of the equivalent homogeneous material. The problem obtained will be like:

$$\begin{cases} \dfrac{d^{\alpha}}{dx^{\alpha}}\left[\hat{K}\dfrac{d^{\alpha}u^{\varepsilon}}{dx^{\alpha}}\right] = -1,\ x \in (0,1) \\ u^{\varepsilon}(0) = u^{\varepsilon}(1) = 0 \end{cases}, \tag{27}$$

that is, it's possible evaluate the effect of apply the conformable derivative indirectly, by put the fractional operator in the Homogenized Problem (Eq 13). In that way:

$$v_0(x) = \frac{2}{\alpha^2\sqrt{15}}\left[x^{\alpha} - \left(x^{\alpha}\right)^2\right], \alpha \in (0,1), \tag{28}$$

which preserve the shape of v_0 in Eq. (26). Moreover, v_0 in Eq. (28) converges to v_0 in Eq. (26), when $\alpha \to 1$, a property of the conformable derivative. In the Fig. 5, these solutions are compared for some values of α, where the effect of applying this operator can be observed: the solution is no more a parabola, even the general shape is looking

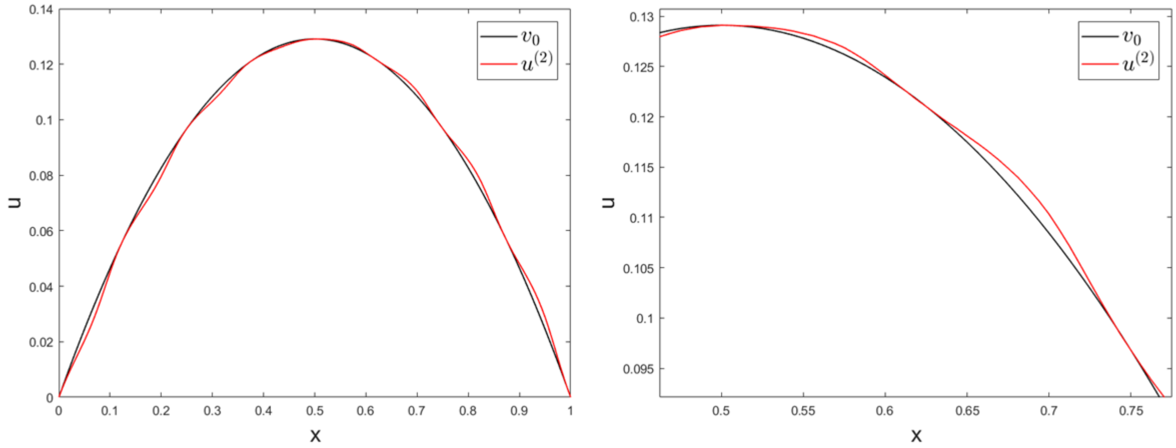

Figure 3: Graphic illustration of the solution $v_0(x)$ and the asymptotic approximation $u^{(2)}$ for $\varepsilon = 1/8$. In the right, the local oscillations of $u^{(2)}$ are highlighted.

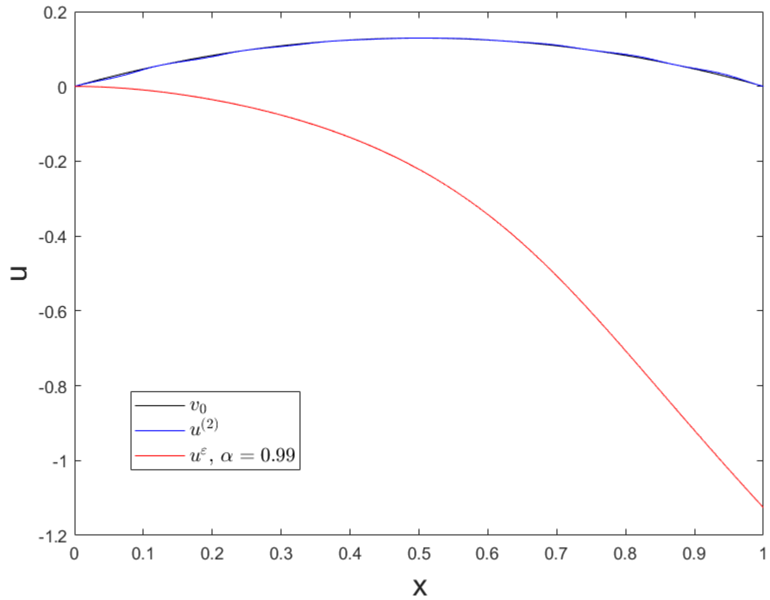

Figure 4: Graphic illustration of the solution u^ε, considering the conformable derivative with $\alpha = 0.99$, compared to v_0 and $u^{(2)}$ (for $\varepsilon = 1/8$).

like. In another words, from the application of the fractional operator, the parabola is deformed and lost its symmetry. If the problem was about a thermal diffusive phenomenon, would be possible make two statements: for the integer order operator, the highest value of temperature is in the middle of the bar; for $\alpha \to 1$, this point isn't in the middle, suggesting a change in the bar effective behaviour. However, the boundary conditions are satisfied for all values of α that are tested (in fact, it can be verified analytically). This type of effect has been observed in others applications of this non-integers operators [10, 11]. In that way, these results suggest that these tools are capable of reproduce some behaviours which the integer order operators aren't. The main idea is: these behaviours are consequence of dissipation nature of the phenomena, like the friction or air resistance.

Besides that, it's possible take the local oscillations with the asymptotic expansions $u^{(2)}$ into the fractional case,

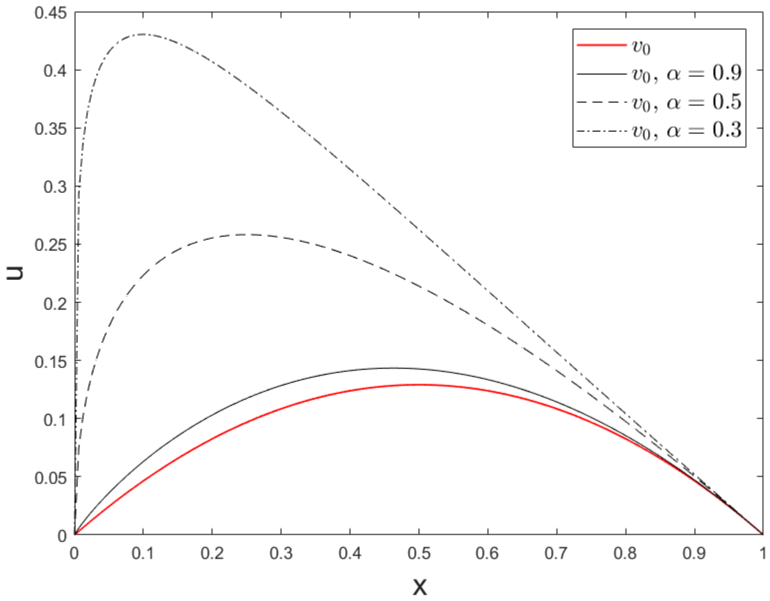

Figure 5: Graphic comparison between $v_0(x)$ from the integer derivative BVP and the non-integer one, for some values of $\alpha \in (0, 1)$.

namely:

$$u_\alpha^{(2)}(x, y) = \frac{2}{\alpha^2\sqrt{15}}\left[x^\alpha - (x^\alpha)^2\right] + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y), \alpha \in (0, 1). \tag{29}$$

In the Fig. 6 a comparison is made between $u_\alpha^{(2)}$ and $u^{(2)}$, and is interesting to notice that as small is the value of α, less significant are the local oscillations that are reproduced by the asymptotic therms $u_1(x, y)$ and $u_2(x, y)$.

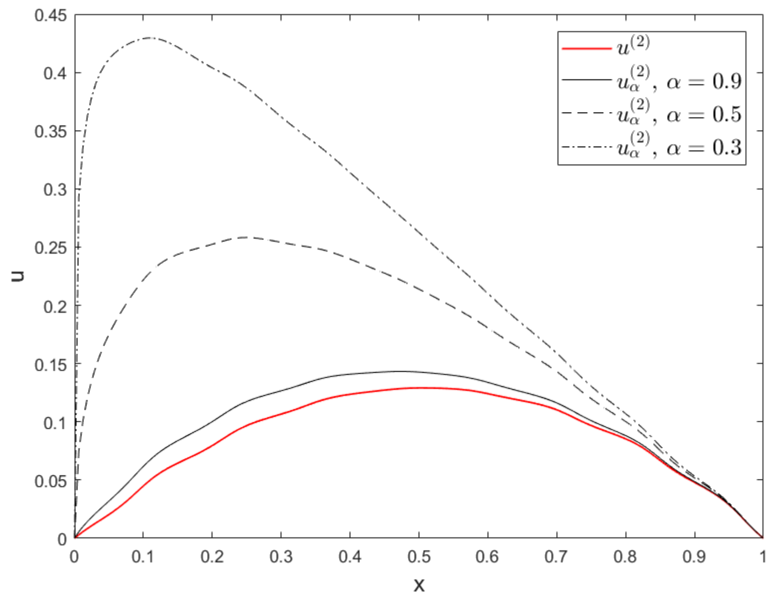

Figure 6: Graphic comparison between $u^{(2)}(x)$ from the integer derivative BVP and the non-integer one, for some values of $\alpha \in (0, 1)$.

With the purpose to expand the examples, the same procedure illustrated in Fig. 5 was applied into a BVP like the one in Eq. (24), but considering another boundary conditions, which are:

$$\begin{cases} \dfrac{d}{dx}\left[K^{\varepsilon}(x)\dfrac{du^{\varepsilon}}{dx}\right] = -1, \ x \in (0,1) \\ u^{\varepsilon}(0) = 0 \ u^{\varepsilon}(1) = 1 \end{cases}, \tag{30}$$

and the result from take the equivalent homogeneous material (with $\hat{K}$) and the conformable derivative is in the Fig. 7. Looking at that, are observed the same effects of considering the fractional derivative in the problem: the solution presents some deformation in relation to that one from the integer order problem, and although the decreasing of the values of α, the boundary conditions still satisfied.

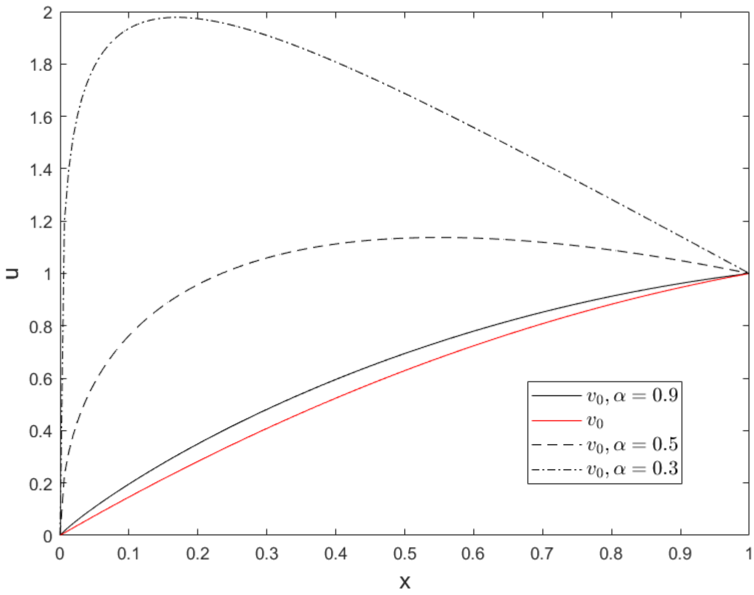

Figure 7: Graphic comparison between $v_0(x)$ from the integer derivative BVP and the non-integer one, for some values of $\alpha \in (0,1)$, with non null boundary conditions.

For final remarks, the behaviour presented in Figs. 5, 6 e 7 are useful to understand how the fractional derivatives (the conformable ones, in this work) can affect the solution of a problem. But it's important to say that a deeper analysis is necessary, to allow more accurate conclusions. Follow some questions which are important to be answered from this paper: what aspect of the structure or phenomenon is responsible for the anomalous behaviour that is reproduced by the fractional derivative; what effects will be found with the application of others fractional operators; what can occurs in dynamical problems, in the sense of apply these operator in time (for these, the Homogenization already shows very good results, in respect of multiple scales); among others.

4 Conclusions

The results in this work show some possibilities of association between the AHM and CF, to solve problems for micro-heterogeneous media, the FGM specifically. Each methodology provides a different aspect of the problem in study.

The AHM takes care of approximate the original solution, reproducing local oscillations of that one or by the homogenized solution, and the applying of the conformable operators causes some interesting effects in the solution of the problem, that are not achieved by the AHM in this work. It indicates that these two tools can be needed to model this type of problem in a more general way. Similar results with FC are found in the literature. Otherwise, for the problem considered, the application of the conformable operator was possible only with the constant coefficient provided by the AHM.

The association of these two tools looks very productive, and deserve more attention, in the sense of searching deeper and more diverse approaches, to obtain better and more effective results. It can be archived by varying the fractional operators and the problems type that are considered.

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