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Parameter N Analysis on the Rational-Talbot Algorithm for Numerical Inversion of Laplace Transform☆

Análise do Parâmetro Numérico N na Transformada Inversa de Laplace Segundo o Algoritmo Talbot-Racional

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Abstract

This article investigates the numerical inversion of the Laplace Transform by the Rational-Talbot method and analyzes the influence on the variation of the free parameter N established by the technique when applied to certain functions. The set of elementary functions, for which the method is tested, has exponential and oscillatory characteristics. Based on the results obtained, it was concluded that the Rational-Talbot method is efficient for the inversion of decreasing exponential functions. At the same time, to perform the inversion process effectively for trigonometric forms, the algorithm requires a greater amount of terms in the sum. For higher values of N, the technique works well. In fact, this is observed in inverting the functions transform, that combine trigonometric and polynomial factors. The method numerical results have a good precision for the treatment of decreasing exponential functions when multiplied by trigonometric functions.

Keywords

Laplace Transform • Numerical Inversion • Rational Approximation

Resumo

Neste artigo investiga-se a inversão numérica da Transformada de Laplace pelo método Talbot-Racional e analisa-se a influência da variação do parâmetro livre N, estabelecido pela técnica, quando aplicado a certas funções. O conjunto de funções elementares, para o qual o método é testado, possui características exponencial e oscilatória. Com base nos resultados obtidos, concluiu-se que o método Talbot-Racional é eficiente para a inversão de funções exponenciais decrescentes. No entanto, para realizar o processo de inversão de forma eficaz para formas trigonométricas, o algoritmo requer uma quantidade maior de termos na soma. Para valores mais elevados de N, a técnica funciona bem. Isso é observado, de fato, na inversão das transformadas de funções que combinam fatores trigonométricos e polinomiais. Os resultados numéricos do método possuem boa precisão para o tratamento de funções exponenciais decrescentes quando multiplicados por funções trigonométricas.

Palavras-chave

Transformada de Laplace • Inversão Numérica • Aproximação Racional

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1 Introduction

Laplace Transform is one of the most applied operational methods for solving differential equations: generations of physicists, mathematicians, and engineers have used this feature as a valuable tool to solve various problems. Despite the advantages it offers, it is essential to recognize that this technique has some limitations. The main difficulty is in determining the respective inverse transform. In this context, the numerical approach became a viable alternative. Several authors have published papers proposing modifications for existing methods to indicate how effective these techniques are. For instance: Talbot [1], Murli and Rizzard [2], Abate and Valkó [3], and Weideman [4].

Talbot [1] was the first to propose an approach of numerical inversion methods of the Laplace Transform [5]. This technique has the evaluation of the complex inversion integral as its main principle, besides the Bromwich (β) contour deformation: it is considered an open path around the negative real axis, provided that no singularity of $F(s)$ is crossed by the β deformation. The purpose of the contour deformation is to reduce in magnitude the exponential factor e^{st} in the integrating function. This technique is widely used for being a robust inversion method and for the easy implementation of the algorithm [6].

New approaches have been formulated since the contour deformation was proposed by Talbot [1]. Trefethen et al. [7] show that two numerical quadrature rules: the trapezoidal rule and the midpoint rule are effective for three contours classes, such as parabolas, hyperbolas, and cotangent curves [8]. Dingfelder and Weideman [8] improved the classical Talbot method using a truncated Talbot contour and a faster convergence was achieved.

Generally, the new inversion algorithms present modifications that are related to the optimal choice of parameters and simple contours, as well. The absence of a universal method has stimulated the various studies: computational aspects such as numerical precision and programming complexities are assessed, confirming that none of the approaches is superior in all criteria for all classes of functions [9]. Within this context, in order to assess the applicability of Talbot's method, proposed by Dingfelder and Weideman [8], here referred to as Rational-Talbot, and to analyze the influence of the free parameter N on the numerical inversion, the algorithm is applied in a set of elementary functions (exponential and oscillatory characteristics), where the inverse transform is known. The application of the midpoint rule in the numerical approximation of the integral is also highlighted.

2 Rational-Talbot Method

To calculate the inverse Laplace Transform in Talbot's method [1], the integral is defined by

$$f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{st}ds = \frac{1}{2\pi i}\oint_{\beta} F(s)e^{st}ds, \mathcal{R}e(s) > \gamma_0, \tag{1}$$

where the β contour is a vertical line defined by $s = \alpha + i\omega$, in $(\alpha - i\infty, \alpha + i\infty)$, α has a fixed value for all singularities of the transformation that are to the left of the line $s = \alpha$ parallel to the imaginary axis.

The approach of the Rational-Talbot method consists of deforming the Bromwich [8] contour. By using a truncated Talbot contour rather than the classical contour that goes to infinity in the left half-plane, faster convergence is achieved. The contour deformation can be justified by Cauchy's [10] theorem, provided the contour remains in the domain of analyticity of $F(s)$. Suppose such contour can be parameterized by

$$C : s = s(\theta), -\pi \le \theta \le \pi, \tag{2}$$

where $\mathcal{R}e(\pm\pi) = -\infty$.

The Bromwich integral from Eq. (1) can be expressed as

$$f(t) = \frac{1}{2\pi i}\int_C e^{st}F(s)ds = \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{s(\theta)t}F(s(\theta))s'(\theta)d\theta. \tag{3}$$

The integral from Eq. (3) is approximated by the midpoint rule with uniform spacing $h = \frac{2\pi}{N}$, which yields

$$f(t) \approx \frac{1}{Ni}\sum_{k=1}^{N} e^{s(\theta_k)t}F(s(\theta_k))s'(\theta_k), \quad \theta_k = -\pi + \left(k - \frac{1}{2}\right)h. \tag{4}$$

Considering quadrature rules with a total of $2N$ nodes [8],

$$s(\theta) \approx \frac{N}{t}\zeta(\theta), \zeta(\theta) = -\sigma + \mu\theta\cot(\alpha\theta) + \nu i\theta, -\pi \le \theta \le \pi, \tag{5}$$

and

$$s'(\theta) = \frac{N}{t}(\mu \cot(\alpha\theta) - \mu\alpha\theta \csc^2(\alpha\theta) + \nu i). \quad (6)$$

In the original Talbot contour the α parameter did not appear, i.e., $\alpha = 1$. Some authors follow this suggestion. However, a modification proposed by Dingfelder and Weideman [8] is to consider $0 < \alpha < 1$. The author suggested parameter choice of σ, μ, ν, and α parameters in Eq. (5). The present work follows this approach.

3 Results and Discussions

In this section, we present and discuss the Rational-Talbot algorithm results, obtained after its implementation.

3.1 Testing functions

Table 1 shows the four elementary functions used to carry out tests and analysis of the inversion method. These tests have the purpose of analysing the applicability of the numerical technique to the inversion of different functions classes and validation of the implemented algorithm. For this, Octave v.5.2.0 open source software on a computer with Windows 7 Ultimate operating system, with 4.0 GB memory (RAM), and AMD E1-1500 APU with Radeon(tm)HD was used during the research.

Taking into account the computational time elapsed, it can be said that the algorithm has good and agile applicability, since, for $N = 18$ and $N = 210$, the lowest and highest value for the N discussed in this work, the execution time on the computer used was 0.55 and 1.35 seconds, respectively.

Table 1: Laplace Transform for elementary functions.

$F(s)$	$f(t)$
$F_1(s) = \dfrac{1}{s+0.5}$	$f_1(t) = e^{-0.5t}$
$F_2(s) = \dfrac{2}{s^2+4}$	$f_2(t) = \sin(2t)$
$F_3(s) = \dfrac{s^2-1}{(s^2+1)^2}$	$f_3(t) = t\cos(t)$
$F_4(s) = \dfrac{1}{s^2+s+1}$	$f_4(t) = \dfrac{2\sqrt{3}}{3}e^{-0.5t}\sin\left(\dfrac{\sqrt{3}}{2}t\right)$

The results are graphically represented by comparing analytical inverse transform and numerical inverse transform. Tables 2-5 present the absolute errors and the values of the free parameter N, which represents the number of terms in the summation for the approximation of $f(t)$. Several tests were carried out for each function, using different values of this parameter to investigate the influence of N in obtaining satisfactory results with good precision. In this work, it was agreed that a satisfactory result presents absolute errors, at most, of 10^{-2} order.

Initially, values indicated by the literature [4, 7, 8], as $N = 18$, $N = 24$ and $N = 32$, were tested. Finally, values greater than $N = 32$ were explored to indicate those for which the method presents satisfactory results.

The comparative study was established based on the results generated by the Rational-Talbot method with fixed values for the parameters $\sigma = 0.6122$, $\mu = 0.5017$, $\nu = 0.2645$, and $\alpha = 0.6407$.

3.2 Inversion of $F_1(s)$

In the inversion of $F_1(s)$, as shown in Table 2 and Fig. 1, with $N = 18$ the Rational-Talbot technique provides excellent results and, with good numerical precision, results in absolute errors of order 10^{-11} and 10^{-12}.

By increasing the value of N, better applicability of this technique is observed. For $N = 24$ the error in the approximation of the exact solution is reduced, showing absolute errors of order 10^{-14} for the two initial instants, and 10^{-15} for the following instants. For $N = 32$ there is an absolute error predominantly of order 10^{-14}.

The inversion method provides accurate results by raising the number of terms in the summation to $N = 50$, with absolute errors of 10^{-13}. From this value onward, the absolute errors become greater, still satisfactory up to $N = 180$, a value where the results show absolute errors varying between 10^{-3} and 10^{-7}.

Starting from $N = 180$, the errors increase and the results deviate from the analytical solution.

Thus, even a greater number of terms in the summation for the approximation is considered, there is no significant increase in computational time.

Table 2: Absolute error of the $F_1(s)$ inversion.

t	$N = 18$	$N = 24$	$N = 32$	$N = 50$	$N = 180$
10^{-4}	5.55124×10^{-11}	1.90958×10^{-14}	8.16013×10^{-14}	7.95141×10^{-13}	1.22822×10^{-3}
1	1.76022×10^{-11}	1.23234×10^{-14}	7.10542×10^{-14}	8.88733×10^{-13}	2.57927×10^{-4}
2	2.53052×10^{-11}	1.16573×10^{-15}	6.79456×10^{-14}	8.05688×10^{-13}	2.80697×10^{-4}
3	3.01191×10^{-11}	5.77315×10^{-15}	6.00908×10^{-14}	6.85007×10^{-13}	4.24368×10^{-6}
4	9.60964×10^{-12}	4.88498×10^{-15}	5.98965×10^{-14}	6.93722×10^{-13}	1.57152×10^{-4}
5	1.33855×10^{-11}	2.26207×10^{-15}	5.53029×10^{-14}	7.14234×10^{-13}	1.29955×10^{-3}
6	3.18867×10^{-11}	4.86416×10^{-15}	5.32976×10^{-14}	6.55739×10^{-13}	5.62179×10^{-5}
7	5.43039×10^{-12}	7.86870×10^{-15}	5.14796×10^{-14}	5.72063×10^{-13}	9.33083×10^{-4}
8	1.96197×10^{-11}	3.23005×10^{-15}	4.79685×10^{-14}	6.06452×10^{-13}	1.27127×10^{-4}
9	1.07390×10^{-11}	5.04110×10^{-15}	4.66068×10^{-14}	5.56053×10^{-13}	5.39632×10^{-7}
10	1.98574×10^{-11}	4.76788×10^{-15}	4.52008×10^{-14}	5.70154×10^{-13}	1.08556×10^{-3}

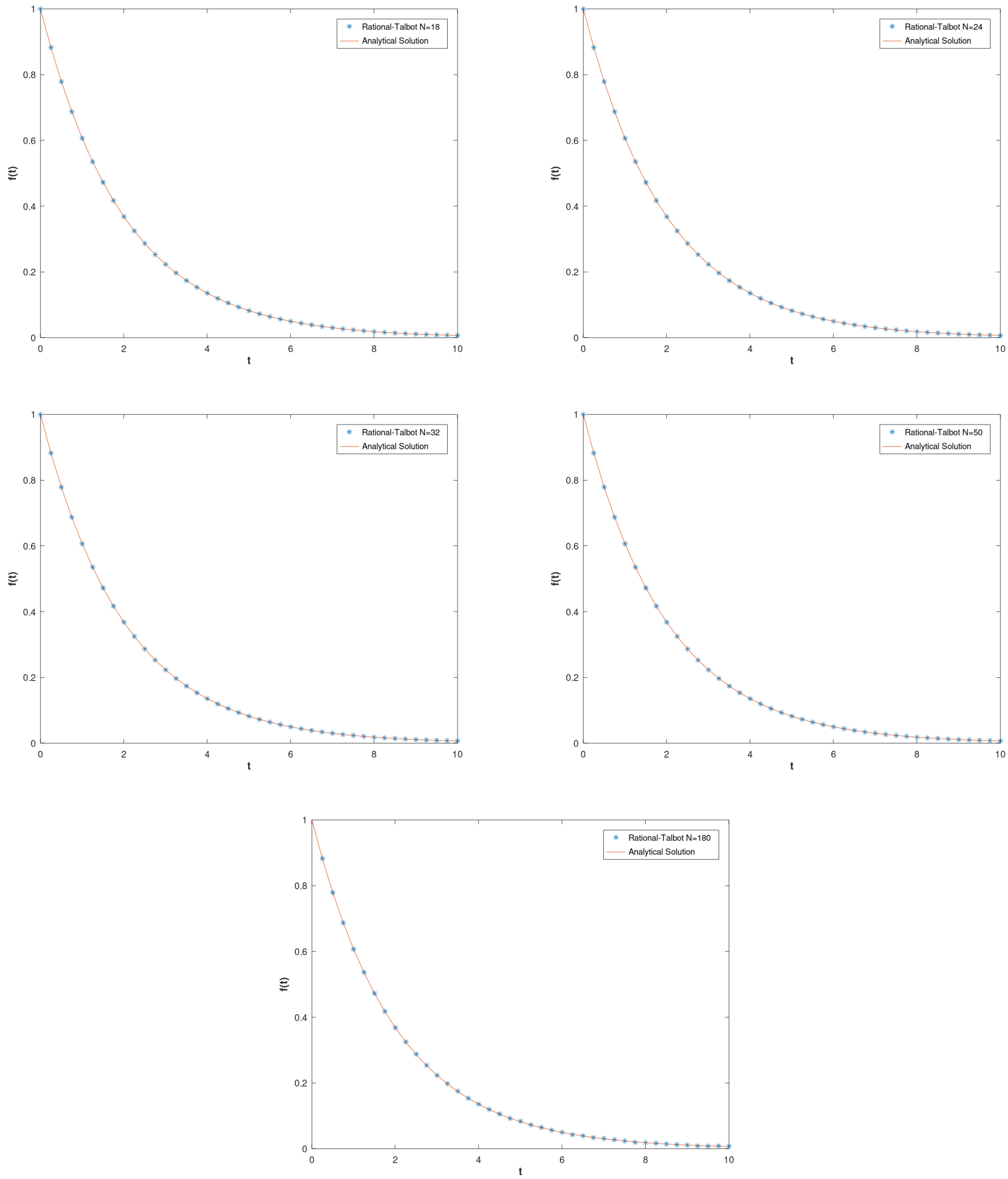

Figure 1: Comparison of the Rational-Talbot method for $F_1(s)$.

3.3 Inversion of $F_2(s)$

Table 3 and Fig. 2 show the numerical inversion tests for the $F_2(s)$ function. It is noticed that the technique is not efficient in inverting functions with oscillatory behavior and a reduced number of terms: a wide numerical instability is evidenced in the graph in Fig. 2. For $N = 18$ the absolute errors oscillate between orders 10^{-13} and 10^{0}.

Adding more terms, $N = 24$ and $N = 32$, there is no better applicability of the technique. It presents a satisfactory approximation compared to the exact solution from $N = 64$, and for this value the absolute errors range from 10^{-16} to 10^{-2}.

The method provided acceptable results up to $N = 190$, where there is an absolute error of order 10^{-8} for the first instant, and as t increases, the absolute errors steadies at order 10^{-3}. From this value, unsatisfactory applicability of the inversion technique can be seen.

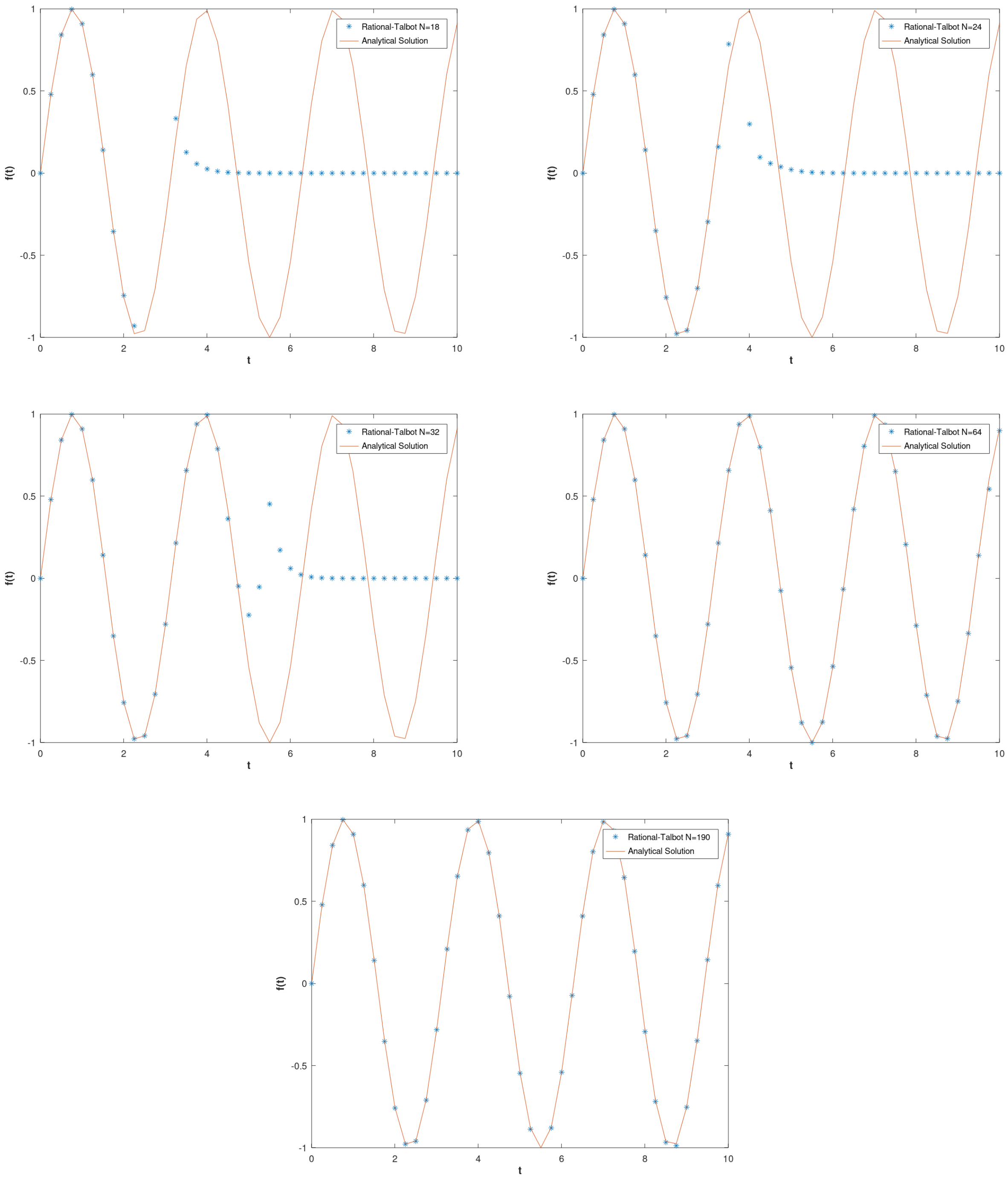

Figure 2: Comparison of Rational-Talbot method for $F_2(s)$.

Table 3: Absolute error of the $F_2(s)$ inversion.

t	$N = 18$	$N = 24$	$N = 32$	$N = 64$	$N = 190$
10^{-4}	1.13219×10^{-13}	4.48215×10^{-17}	2.23616×10^{-19}	1.07698×10^{-16}	1.00413×10^{-8}
1	6.99579×10^{-6}	1.53707×10^{-7}	5.33404×10^{-11}	1.06820×10^{-11}	7.12507×10^{-4}
2	1.14825×10^{-2}	1.00589×10^{-4}	1.37125×10^{-7}	1.97515×10^{-11}	1.36974×10^{-3}
3	2.07481×10^{0}	1.66575×10^{-2}	4.41917×10^{-5}	2.67643×10^{-11}	2.02972×10^{-3}
4	9.63822×10^{-1}	6.90514×10^{-1}	5.51945×10^{-3}	3.02134×10^{-11}	2.76653×10^{-3}
5	5.44755×10^{-1}	5.65335×10^{-1}	3.20423×10^{-1}	3.73712×10^{-11}	2.30380×10^{-3}
6	5.36567×10^{-1}	5.37602×10^{-1}	5.96833×10^{-1}	2.56183×10^{-9}	3.78582×10^{-3}
7	9.90613×10^{-1}	9.90574×10^{-1}	9.90046×10^{-1}	4.15239×10^{-7}	6.95537×10^{-3}
8	2.87902×10^{-1}	2.87902×10^{-1}	2.87832×10^{-1}	4.02757×10^{-5}	5.09579×10^{-3}
9	7.50986×10^{-1}	7.50986×10^{-1}	7.50979×10^{-1}	2.16334×10^{-3}	2.22158×10^{-3}
10	9.12945×10^{-1}	9.12945×10^{-1}	9.12945×10^{-1}	1.36130×10^{-2}	4.14141×10^{-3}

3.4 Inversion of $F_3(s)$

As for the inversion of $F_3(s)$, as can be seen in Table 4 and Fig. 3, for the values indicated in the literature, the method does not present good applicability for the inversion of functions involving a product of polynomial and trigonometric factors.

The inversion using the Rational-Talbot method with $N = 18$ results in an absolute error of order 10^{-14} for the initial instant, for the interval $t = [1, 4]$, absolute errors between 10^{-6} and 10^{-1}, and of 10^{0} for the following instants, except for $t = 6$ and $t = 8$.

It is observed that the results deviate considerably from the analytical solution for the instants $t = [5, 10]$. For values $N = 24$ and $N = 32$, the method behaves similarly, showing a better approximation for $N = 32$.

From $N = 38$, there are satisfactory results, absolute errors ranging between 10^{-19} and 10^{-2}. The method shows acceptable results up to $N = 210$, starting with absolute errors of order 10^{-8} and ending in 10^{-2}. After this value, the results deviate from the analytical solution.

Table 4: Absolute error of the $F_3(s)$ inversion.

t	$N = 18$	$N = 24$	$N = 32$	$N = 38$	$N = 210$
10^{-4}	5.66096×10^{-14}	2.24141×10^{-17}	1.05032×10^{-19}	1.32137×10^{-19}	3.91428×10^{-8}
1	5.26717×10^{-6}	7.96161×10^{-9}	9.22817×10^{-13}	1.60982×10^{-14}	1.11365×10^{-3}
2	4.45805×10^{-4}	3.13004×10^{-7}	4.65338×10^{-10}	1.36790×10^{-12}	1.88326×10^{-3}
3	1.87773×10^{-2}	9.71377×10^{-5}	6.22009×10^{-8}	1.50071×10^{-11}	4.88038×10^{-3}
4	2.85804×10^{-1}	2.83849×10^{-3}	2.57867×10^{-6}	1.23378×10^{-8}	4.06071×10^{-3}
5	2.86565×10^{0}	4.36566×10^{-2}	3.09477×10^{-5}	3.14633×10^{-7}	4.93728×10^{-2}
6	6.45973×10	5.00498×10^{-1}	1.27082×10^{-4}	2.46652×10^{-6}	2.38822×10^{-3}
7	6.78409×10^{0}	6.79802×10^{0}	1.96847×10^{-4}	1.50185×10^{-5}	9.89466×10^{-3}
8	8.31551×10^{-1}	6.41935×10^{0}	9.42173×10^{-3}	5.52510×10^{-4}	3.16175×10^{-4}
9	8.12604×10^{0}	7.77443×10^{0}	2.98689×10^{-1}	6.00475×10^{-3}	6.05298×10^{-2}
10	8.37650×10^{0}	8.11264×10^{0}	7.49848×10^{0}	2.44193×10^{-2}	7.19238×10^{-2}

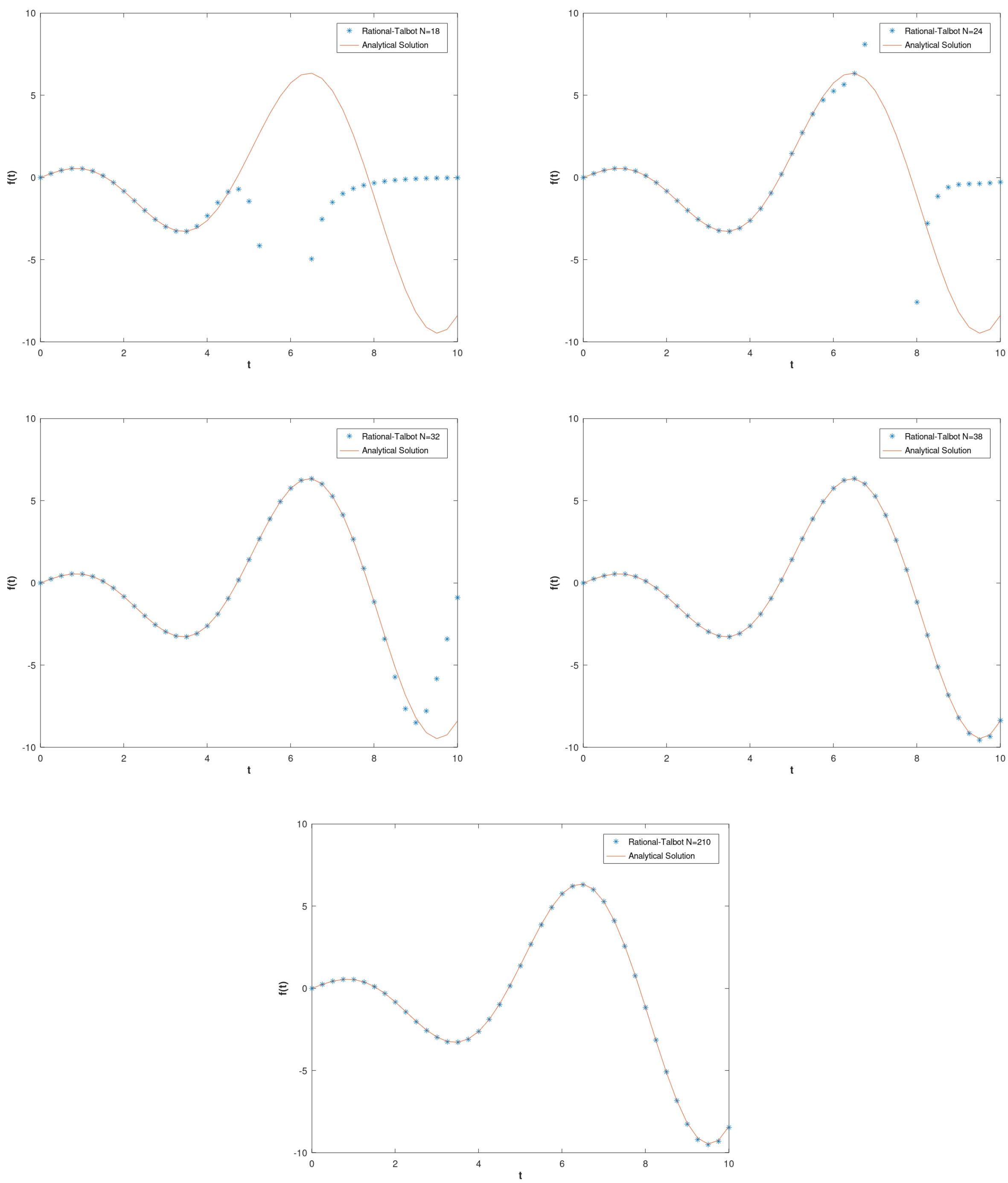

Figure 3: Comparison of the Rational-Talbot method for $F_3(s)$.

3.5 Inversion of $F_4(s)$

Analyzing Table 5 and Fig. 4, for inversion of functions of the same class as $F_4(s)$, involving an exponential multiplication result with oscillation, the Rational-Talbot technique provided a satisfactory precision. With $N = 18$, it produced absolute errors of order 10^{-14} and 10^{-3}. For $N = 24$, there is an absolute error varying between 10^{-17} and 10^{-5}. Better applicability of this technique is observed for $N = 32$, where the absolute errors oscillate between 10^{-19} and 10^{-8}.

By increasing the value of N, the inversion method provides good results up to $N = 50$, presents absolute errors of 10^{-19} order for the initial instant, 10^{-14} for $t = 1$, and 10^{-13} for the following instants. From this value onward, the absolute errors become greater until $N = 190$, where they vary between 10^{-9} and 10^{-3}. After $N = 190$, the results deviate from the analytical solution, resulting in unsatisfactory applicability of the inversion technique.

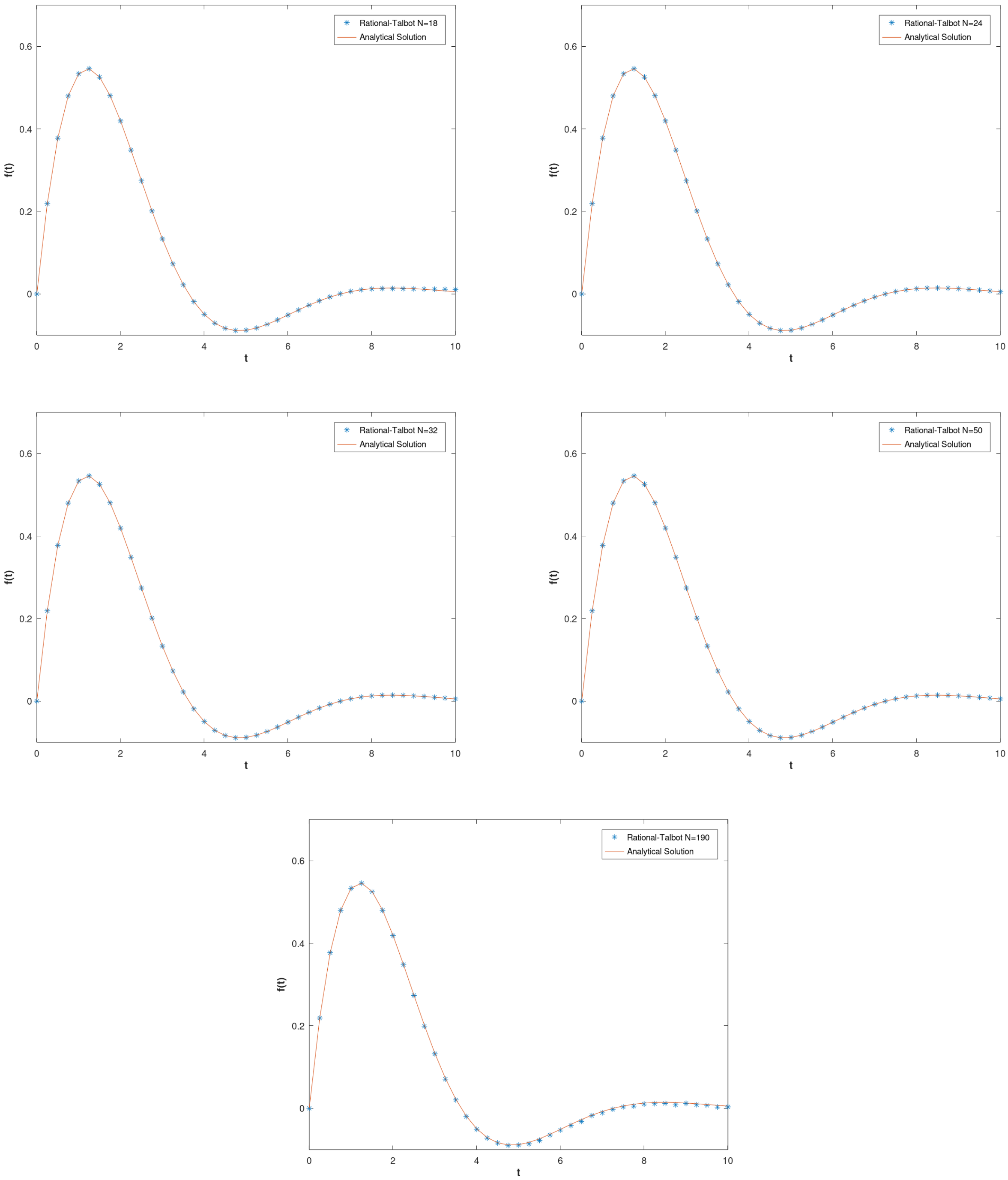

Figure 4: Comparison of the Rational-Talbot method for $F_4(s)$.

Table 5: Absolute error of the $F_4(s)$ inversion.

t	$N = 18$	$N = 24$	$N = 32$	$N = 50$	$N = 190$
10^{-4}	5.65991×10^{-14}	2.24040×10^{-17}	1.08420×10^{-19}	8.08069×10^{-19}	5.04782×10^{-9}
1	2.13226×10^{-8}	2.14920×10^{-11}	6.66133×10^{-15}	7.33857×10^{-14}	3.33759×10^{-4}
2	1.35424×10^{-7}	5.09335×10^{-10}	3.53606×10^{-14}	1.29563×10^{-13}	6.49540×10^{-4}
3	3.69815×10^{-6}	1.04562×10^{-8}	1.99326×10^{-12}	1.34253×10^{-13}	9.55364×10^{-4}
4	2.10386×10^{-5}	5.44210×10^{-8}	1.36557×10^{-11}	2.03254×10^{-13}	1.28441×10^{-3}
5	7.34205×10^{-5}	9.35699×10^{-8}	1.59235×10^{-10}	2.39946×10^{-13}	1.00813×10^{-3}
6	2.04600×10^{-4}	3.19940×10^{-8}	5.42230×10^{-10}	2.10761×10^{-13}	1.81914×10^{-3}
7	4.31495×10^{-4}	9.74553×10^{-8}	9.05862×10^{-10}	2.36382×10^{-13}	3.15305×10^{-3}
8	4.84274×10^{-4}	2.47957×10^{-6}	4.46986×10^{-10}	2.49194×10^{-13}	2.24617×10^{-3}
9	5.76268×10^{-4}	1.51737×10^{-5}	2.08846×10^{-10}	2.64564×10^{-13}	8.71704×10^{-4}
10	4.99630×10^{-3}	5.42285×10^{-5}	1.69937×10^{-8}	2.57624×10^{-13}	1.82784×10^{-3}

4 Conclusions

In this work, the Rational-Talbot method was presented, as well as validation of the algorithm to invert four elementary functions with exponential and oscillatory characteristics. The obtained results through the tests were assessed and compared to the analytical inverse transforms. The free parameter N influence, specified by the technique on the numerical precision of the results, was also investigated.

The method applicability analysis in the inversion of each function allowed to make recommendations about its use: Rational-Talbot method presents excellent applicability for the treatment of $F_1(s)$ since it did not require a large number of terms in the series and provided satisfactory absolute errors from $N = 18$. From the investigation of the parameter N influence, it is noticed that it produces smaller absolute errors for $N = 24$ to $N = 50$. From this value, the higher value of N, the more precision is lost in the approximation by this method.

For the inversion of $F_2(s)$, it was essential to increase the number of terms in the summation to ensure greater numerical precision. The method fails for values of N indicated in the literature. From $N = 64$ onward, a good approximation was obtained when compared to the exact solution. Good results are shown until $N = 190$.

When it comes to $F_3(s)$, the Rational-Talbot method is not efficient for a smaller number of terms in the series, but there was a better applicability for larger values of N. Between $N = 38$ and $N = 210$, the technique presents satisfactory results.

In the inversion of $F_4(s)$, the algorithm proved to be accurate when applied to invert this type of function. It presented results with less error, with a reduced number of N. It provides good approximations up to $N = 190$.

In the analysis of this work, it is concluded that the Rational-Talbot method is more suitable for cases where the inverse transform is a decreasing exponential function ($F_1(s)$) or a decreasing exponential function multiplied by a trigonometric function ($F_4(s)$). In the tests carried out to invert the transform of trigonometric functions, functions with oscillatory behavior ($F_2(s)$ and $F_3(s)$), the algorithm presents good applicability, however, it requires a considerable amount of terms in the summation to perform the inversion process effectively. It is highlighted that the number of summation terms for the numerical approximation of $f(t)$, through the Rational-Talbot method, depends on the function and its inverse form. In addition, when the number of this parameter is significantly increased, the method does not present good results.

For future studies, it is suggested to investigate the application of the Rational-Talbot method for the Laplace Transform numerical inversion of functions with different behaviors, along with exponential and oscillatory behavior, and to study other methods applicable to the numerical analysis of the inverse Laplace Transform.

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